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Asymptotic solutions of Lindblad equations

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Abstract
The temporal motion of observables of a quantum mechanical N-level system is studied. In particular, I investigate the mapping, in its dependence on the matrix V parametrizing the Lindblad generator, of given initial configurations into the resulting configurations at large times t ($t \to \infty$). Explicit solutions are given for a large class of V.

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1. Introduction

The description of a physical system requires an operational definition of its state and of the state changes produced by external manipulations. Very general settings for state changes in quantum systems have been discussed by Haag and Kastler [1] in the framework of algebraic quantum theory. In the common formulation of quantum theory in Hilbert space, two scenarios are of particular importance. States described as density operators change, in the first scenario, when the system under consideration, in the state ϱ, interacts with a second system ('apparatus'), in the state ϱ'; a measurement of a property a' of the apparatus corresponding to a projection operator $P_{a'}$ leaves the system in the state [2–4]

$$\tilde{\varrho} = (\mathbb{I} \otimes P_{a'})U^{+}(\varrho \otimes \varrho')U(\mathbb{I} \otimes P_{a'}) \tag{1.1}$$

where U stands for the unitary motion of the total system.

The map

$$T : \varrho \mapsto \tilde{\varrho} = T(\varrho)$$

is shown [5] to be of the form

$$\tilde{\varrho} = \sum_{i \in J} Q_i \varrho Q_i^{+} \tag{1.2}$$

where J is some index set and

$$\sum_{i \in J} Q_i^{+} Q_i \leqslant \mathbb{I}. \tag{1.3}$$

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Furthermore, it can be readily seen [5] that its adjoint map T^+ which describes the state change in the Heisenberg picture

$$T^+ : \tilde{B} = \sum_{i \in J} Q_i^+ B Q_i \qquad B, \tilde{B} \in \mathfrak{B}(\mathfrak{H}) \quad \text{norm-bounded observables on } \mathfrak{H} \tag{1.4}$$

is a completely positive map. The Stinespring theorem [6] then leads [2] to the lemma that a map $\mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H})$ is completely positive if it can be written as (1.4).

The second scenario encompasses situations in which the systems under consideration are in 'contact' with a bath. The map

$$\begin{aligned} \hat{T} : \mathfrak{B}(\mathfrak{H}) &\mapsto \mathfrak{B}(\mathfrak{H}) \\ B &\mapsto \tilde{B} \\ \tilde{B} &= \mathrm{Tr}_{\mathrm{bath}}(\varrho_{\mathrm{bath}} U^+ (B \otimes \mathbb{I}) U) \end{aligned} \tag{1.5}$$

is now obtained by averaging over the bath variables; the map $\hat{T}$ is, obviously, again completely positive.

The most general basis for the formulation of equations describing the time evolution of the system observables (or states) is doubtlessly the *ab initio* approach which starts from the unitary motion of the combined system plus bath variables. A remarkable broad corpus of models has been developed [7, 8] which in a general sense sets the scope within which the motion of system observables or states is effected by a semigroup of completely positive maps, the most important issues being the existence of Markovian equations of motion and the quantification of time scales comparing particular scales for the motion of the embedded quantum system and the relaxation times in the bath [9–12].

Complete positivity and the semigroup structure then lead to the celebrated Lindblad equations [13]. Needless to say but important to keep in mind that the derivation of the latter follows an axiomatic line and there is no *a priori* answer to the question whether or not and, if yes, under which circumstances these equations describe the dynamics of open systems. One of the pungent questions is how the operators V_i parametrizing the generator of temporal motions are to be chosen to describe a specific experimental situation. More precisely, we have

$$\begin{aligned} \dot{B} = \mathrm{i}[H, B] + \sum_{i \in J} V_i^+ B V_i - \frac{1}{2}[V_i^+ V_i, B]_+ \\ V_i, \sum_{i \in J} V_i^+ V_i \in \mathfrak{B}(\mathfrak{H}) \quad \text{norm-bounded operators on } \mathfrak{H} \end{aligned} \tag{1.6}$$

(in the following we shall work in the Heisenberg picture), where J is an (eventually continuous) index set.

In principle, the V_i should be determined from the dynamics imposed by the experimental set-up—a formidable task indeed. A more down-to-earth approach certainly is to try to model whatever process is under consideration by making physically plausible ansätze for these operators V.

In this paper, we pose a more modest question: can we classify asymptotic (as time goes to infinity) configurations of observables B according to specific properties—symmetries—of Lindblad operators V_i? We pursue, to begin with, this question in the special situation in which only one V_i prevails. Situations which require more than one operator can be treated after a rather general answer is found for the case of one operator.

2. Lindblad damping in *N*-level systems with privileged *V*

We choose to work in the Heisenberg picture and discuss the asymptotic behaviour for $t \to \infty$ of solutions of the Lindblad equation $\left(\dot{A} = \frac{\mathrm{d}}{\mathrm{d}t}A\right)$:

$$\dot{A} = \mathrm{i}[H, A] + V^{+}AV - \tfrac{1}{2}[V^{+}V, A]_{+} \tag{2.1}$$

where H denotes the Hamiltionian of the system, A an observable and $[H, A] = HA - AH$, $[\cdots]_{+}$ the commutator and anticommutator respectively; H and A are Hermitian, V are arbitrary $N \times N$ complex matrices of rank N representing linear operators on $\mathfrak{H}$ (or $\mathbb{C}^N$). We assume rank $(H, A, \ldots) = N$. To study pure Lindblad damping, we put in this section $H = 0$. The equation of motion then reads

$$\begin{aligned} &\dot{B} = V^{+}BV - \tfrac{1}{2}[V^{+}V, B]_{+} \\ &B = H, A, \ldots . \end{aligned} \tag{2.1$'$}$$

The problem we now pose is to explore the dependence on the Lindblad matrix V of asymptotic configurations ($t \to \infty$) of observables B, an initial configuration given at, say, $t = 0$. By 'configuration' we mean a specific matrix obtained from the equation of motion at a specific time t_1: the cross section of the flow obtained from (2.1′) at $t = t_1$.

Seen more formally, we intend to examine general properties of the mapping

$$\begin{aligned} &\tau(V) : \mathbb{C}^{N^2} \to \mathbb{C}^{N^2} \\ &B|_{t=0} \mapsto B_{t=\infty} \end{aligned} \tag{2.2}$$

for solutions B of (2.1′). In particular, we try to find classes of V, as far as possible to emphasize their relevance, leading to typical asymptotic behaviour.

To approach this problem, we use a more hermeneutic method of solution: we impose general properties of Lindblad matrices V with increasing complexity and derive the corresponding asymptotic configurations for general initial conditions.

2.1. General Lindblad matrices

The simplest case to be considered is as follows:

(i) V is Hermitian,

$$[B, V] = 0 \qquad \text{for all} \quad t$$

which is consistent but trivial since then

$$\dot{B} = 0$$

and B remains in its initial state: $\tau(V) = \mathrm{id}$. Thus we consider

$$[B, V] \neq 0$$

except, possibly, at some discrete values of $t > 0$.

Assuming Hermiticity of V means that we may choose a basis in $\mathfrak{H}$ in which V is diagonal

$$V_{\mathrm{diag}} = \begin{pmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_N \end{pmatrix} \qquad \nu_i \in \mathbb{R}, i = 1, \ldots, N.$$

Denote

$$B = (b_{ij}(t))$$

(2.1′) now reads

$$\dot{b}_{ik}(t) = \nu_i \nu_k b_{ik}(t) - \frac{1}{2}\left(\nu_i^2 + \nu_k^2\right) b_{ik}(t)$$

and has the solution

$$b_{ik}(t) = b_{ik}(0)\,\mathrm{e}^{-\frac{1}{2}(\nu_i - \nu_k)^2 t}. \tag{2.3}$$

The map $\tau(V))$ is now easily discussed: in the case of no degeneracy, i.e. all ν_i are different, $\tau(V)$ maps all initial configurations on their diagonal part,

$$\tau(V) : (b_{ik}(0)) \longmapsto \begin{pmatrix} b_{11}(0) & & 0 \\ & \ddots & \\ 0 & & b_{NN}(0) \end{pmatrix}. \tag{2.4}$$

In the case of degeneracies, the corresponding off-diagonal elements remain.

A general Hermitian V can be written as

$$V = U V_{\mathrm{diag}} U^{-1} \tag{2.5}$$

where U is unitary. We immediately see that now

$$\tau(V) : U^{-1} B|_{t=0} U \longmapsto U^{-1} B_{t=\infty} U|_{\text{‘diagonal’}} \tag{2.6}$$

‘diagonal’ includes the possibility of degeneracies in V. This relation specifies the map $\tau(V)$ for all Hermitian V for any given initial configuration.

We now turn to the second part of the alternative.

(ii) V is non-Hermitian. Needless to say that this notion is a bit cursory or even superficial since after all ‘almost all’ matrices in $\mathbb{C}^{N^2}$ are non-Hermitian. As already indicated, our strategy will be to discuss certain classes of matrices with well-defined general properties and study the repercussions of these properties on the asymptotic solutions of (2.1′).

At the centre of these constructions is the immediate observation that V^+V is Hermitian and positive (we demanded maximal rank for V) so that a basis in $\mathfrak{H}$ can be chosen such that V^+V is diagonal with positive entries. We take V^+V as the starting point of our construction. The problem of determining V is then similar to constructing ‘square roots’ of diagonal matrices, a problem for which the positivity of V^+V is of prime importance.

We then note that V^+V is positive and diagonal iff V is built up by N linearly independent and, moreover, orthogonal column vectors $\vec{\nu}_i$ $i = 1, \ldots, N$ with components

$$\vec{\nu}_i = (\nu_{ki})\nu_{ki} \in \mathbb{C} \qquad i, k = 1, \ldots, N$$

and

$$\vec{\nu}_i^+ \vec{\nu}_k = |\vec{\nu}_i|^2 \delta_{ik}. \tag{2.7}$$

2.2. *A general solution for symmetry inducing Lindblad matrices V*

We start by choosing a basis in which V^+V is diagonal

$$V^+V = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_N \end{pmatrix} w_i > 0 \qquad i = 1, \ldots, N$$

and construct a matrix V— a set of orthogonal vectors $\vec{\nu}_i$— which reflects a situation in which relevant symmetry leads to characteristic asymptotic behaviour of observables. Considerable simplification occurs if we demand that these vectors coincide with the coordinate axis chosen for representing V^+V.

We put

$$\begin{aligned} \nu_{ki} &= \nu_i \delta_{kfL(i)} \qquad \nu_i \in \mathbb{C} \qquad i = 1, \ldots, N \\ w_i &= |\vec{\nu}_i|^2 \end{aligned} \tag{2.8}$$

where f_L is an arbitrary permutation, a bijective map of a set of N integers, say $[1, \ldots, N]$,

$$f_L : [1, \ldots, N] \to [1, \ldots, N]$$

which fixes the sequence of identification of the $\vec{\nu}_i$ with the axis of the basis. The maps

$$f_L : i \longmapsto f_L(i) = (i + L) \bmod N + 1 \tag{2.9}$$

reveal the important points. To mention it more precisely the cyclic permutations of $(L + 1)$-plets realized by f_L should serve as a paradigmatic example which illustrates the essential cyclic structure.

In the following, we have to go into some details and write down the permutation machinery which is important for us.

We have (f_L^κ denotes $f_L \circ f_L \circ \cdots \circ f_L$, κ times)

$$\begin{aligned} &f_0[1, \ldots, N] = [2, \ldots, N, 1] \qquad f_0^N = \mathrm{id} \\ &f_1[1, \ldots, N] = [3, 4, \ldots, N, 1, 2] \qquad f_1^{N/2} = \mathrm{id} \quad N/2\text{—integer} \\ &\vdots \\ &f_L[1, \ldots, N] = [L + 1, \ldots, N, 1, \ldots, L] \qquad f_L^{N/(L+1)} = \mathrm{id} \quad N/(L + 1)\text{—integer} \\ &\vdots \\ &f_{N-1} = \mathrm{id} \end{aligned}$$

if N is a prime number

$$f_L^N = \mathrm{id} \qquad \text{for all} \quad L \geqslant 0.$$

Hence, if $N/(L + 1)$ is integer the ordered set $[1, \ldots, N]$ is decomposed into $(L + 1)$ adjacent sets which are cyclically permuted by f_L so that we obtain $N/(L + 1)$ permutations of the original set $[1, \ldots, N]$. We then arrange consecutively the first elements of these permutations into a set of length $N/(L+1)$, the second elements into a second and so on until we reach the $(L+1)$th element to get a $(L+1)$th set. In this way we constructed $L+1$ sets of indices $\left[i_{\alpha_1}, \ldots, i_{\alpha_K}\right]$,

$$(L + 1)\ R\text{-cyles of length} \qquad K = N/(L + 1). \tag{2.10}$$

The subspace spanned by the unit vectors $\left[\vec{\nu}_{\alpha_1}, \ldots, \vec{\nu}_{\alpha_N}\right]$ defined in (2.8) is obviously invariant under these cyclic permutations. The point we will prove is that asymptotic configurations are given as a direct sum of scalar multiples of the unit matrices in these invariant subspaces. By construction f_L induces cyclic permutations of these R-cycles.

Inserting (2.8) in (2.1′) we obtain the equations of motion

$$\dot{b}_{li} = \nu_l^* \nu_i b_{fL(l)fL(i)} - \tfrac{1}{2}(|\vec{\nu}_i|^2 + |\vec{\nu}_l|^2) b_{li} \qquad l, i = 1, \ldots, N \tag{2.11}$$

for the $N(N + 1)/2$ functions $b_{li}(t), l \leqslant i$. The first important feature to be read from these equations is that diagonal and off-diagonal elements of B decouple. More precisely, to determine sets of coupled equations we consecutively take the equations coupling indices $(i, l) \to (f_L(i), f_L(l)) \to \left(f_L^2(i), f_L^2(l)\right) \to \left(f_L^K(i), f_L^K(l)\right) = (i, l)$: the indices follow the R-cycles defined above, we get a set of K equations.

Considering now diagonal elements we get N equations or, rather, $(L+1)$ sets of K equations with index pairs to be read from R-cycles:

$$\dot{b}_{ii} + |\nu_i|^2(b_{ii} - b_{fL(i)fL(i)}) = 0 \qquad i = 1, \ldots, K. \tag{2.12}$$

We anticipate the $b_{ii}(t)$ to decrease for $t \to \infty$; this is expected due to the dissipative structure of the Lindblad generator and, in particular, from the assumption of complete positivity used in this derivation. So to understand asymptotic configurations, we have to explore the stationary configurations of the Lindblad equation: (2.11) is a set of linear differential equations with constant coefficients so that exponential decay will result, the components remaining at $t = \infty$ are hence determined by the zero modes.

Let us introduce the following notation:

R-cycles of length K:

$$R^{(\alpha)} = \left[i^{(0)}_{\alpha 1}, \ldots, i^{(0)}_{\alpha K}\right] \qquad \alpha = 1, \ldots, L+1$$

P_c, cyclic permutation of K elements

$$P_c^\gamma = P_c \circ P_c \circ \cdots \circ P_c \qquad \gamma \text{ times.}$$

We obviously have a zero mode for each R-cycle $R^{(\alpha)}$, a fact which we readily read from equation (2.12),

$$\text{zero mode} \quad \dot{b}_{ii} = 0 \qquad i = 1, \ldots, K$$

and

$$\begin{aligned} b_{i_{\alpha 1} i_{\alpha_1}} &= b_{i_{\alpha_2} i_{\alpha_2}} = \cdots = b_{i_{\alpha_K} i_{\alpha_K}} \\ &=: b^{(\alpha)}(\infty). \end{aligned} \tag{2.13}$$

That is to say that all the diagonal elements of $B|_{t=\infty}$ enumerated by an index set $R^{(\alpha)}$, are equal and non-vanishing for any non-trivial initial configuration, the characteristic polynomial of (2.12) thus has a zero mode. To show the non-vanishing of the $b^{(\alpha)}(\infty)$, $\alpha = 1, \ldots, L$, we simply calculate $b^{(\alpha)}(\infty)$ in terms of the initial configuration. The important step is to derive constants of motion for (2.12). Adding appropriately multiplied equations (2.12) we arrive at

$$\sum_{\gamma=1}^{K} \dot{b}_{i^{(\gamma)}_{\alpha 1} i^{(\gamma)}_{\alpha 1}} w_{i^{(\gamma)}_{\alpha 2}} \cdots w_{i^{(\gamma)}_{\alpha 1}} = 0$$

where

$$\left[i^{(\gamma)}_{\alpha 1}, \ldots, i^{(\gamma)}_{\alpha K}\right] := P_c^\gamma \left[i^{(0)}_{\alpha 1}, \ldots, i^{(0)}_{\alpha K}\right].$$

Integrating this relation and equating the result for $t = 0$ and $t = \infty$ we get in view of (2.14)

$$b^{(a)}(\infty) = \frac{\sum_{\gamma=1}^{k} b_{i^{(\gamma)}_{\alpha 1}} i^{(\gamma)}_{\alpha 1}(0) w_{i^{(\gamma)}_{\alpha 2}} \ldots w_{i^{\gamma}_{\alpha K}}}{\sum_{\gamma=1}^{k} w_{i^{(\gamma)}_{\alpha 2}} \ldots w_{i^{(\gamma)}_{\alpha K}}}. \tag{2.14}$$

This result has a very intriguing interpretation: the asymptotic diagonal elements $b^{(\alpha)}(\infty)$ are simply the statistical average of the initial diagonal elements with a normalized probability distribution determined by the matrix elements of the Lindblad matrix: $w_i = |\nu_i|^2$.

As far as off-diagonal elements are concerned we again look for non-trivial solutions of

$$V^+ B\, V - \tfrac{1}{2}[V^+ V, B]_+ = 0. \tag{2.15}$$

Iterating any off-diagonal equation along an R-set we find from (2.11)

$$b_{ei} = c_{ei}\, c_{f_L}(e) f_L(i) \ldots c_{f_L^{K-1}(e) f_L^{K-1}(i)} b_{ei}$$

with

$$c_{ei} := \frac{\nu_e^* \nu_i}{\frac{1}{2}(|\nu_i|^2) + |\nu_e|^2}.$$

Since

$$|c_{ei}| < 1 \qquad \text{if} \quad \nu_e \neq \nu_i$$

we conclude that

$$b_{ei} = 0$$

unless

$$\nu_{f_L^\alpha(e)} = \nu_{f_L^\alpha(i)} \qquad \text{for all} \quad \alpha = 0, \ldots, K-1. \tag{2.16}$$

Thus to obtain a non-vanishing off-diagonal asymptotic matrix element b_{ei} we have to look up the R-cycle(s) containing e and i and equate all ν following these cycles. Clearly, after a few steps all ν_i are equal,

$$\nu_i = \tilde{\nu} \qquad i = 1, \ldots, N \tag{2.17}$$

and the Lindblad matrix is a multiple of a 'square root of unity',

$$V = \nu\sqrt{\mathbb{I}} \tag{2.18}$$

where $\mathbb{I}$ is the $N \times N$ unit matrix. Thus we have $N!$ roots which are to be classified following the method described above where we restricted our discussion to the case of maps f_L defined in (2.9). Letting the latter act on the remaining non-cyclic permutations of $[1, \ldots, N]$ will cover all other cases which, however, do not lead to essentially (see below) new results.

After this digression, let us return to the discussion of off-diagonal elements. Taking now a set of ν_i obeying (2.17) we obtain non-vanishing off-diagonal contributions to $B|_{t=\infty}$ which follow from the same averaging procedure as given in (2.16). Writing down the corresponding general formulae would require either cryptic or, if one attempts to be explicit, clumsy notation. So we think it might be of use to illustrate the general results in an explicit example.

2.3. *Explicit solutions for $N = 6$*

We begin with $L = 0$ and not all ν_i equal, i.e. equations (2.17) are not obeyed and our asymptotic configuration is diagonal or, more precisely, a multiple of the unit matrix,

$$B|_{t=\infty} = b^{(0)}(\infty)\mathbb{I} \tag{2.19}$$

with (see equation (2.15))

$$b^{(0)}(\infty) = \frac{b_{11}(0)w_2w_3w_4w_5w_6 + \cdots + b_{66}(0)w_1w_2w_3w_4w_5}{w_1w_2w_3w_4w_5 + w_1w_2w_3w_4w_6 + \cdots + w_2w_3w_4w_5w_6} \tag{2.20}$$

for any initial configuration.

Take $L = 1$ and hence $K = 3$. We have two R-cycles

$$R^{(1)} = [1, 3, 5] \qquad R^{(2)} = [2, 4, 6].$$

The asymptotic configuration is then

$$B|_{t=\infty} = \begin{pmatrix} b^{(1)}(\infty) & & & & & \\ & b^{(2)}(\infty) & & & 0 & \\ & & b^{(1)}(\infty) & & & \\ & & & b^{(2)}(\infty) & & \\ & & & & b^{(1)}(\infty) & \\ & 0 & & & & b^{(2)}(\infty) \end{pmatrix} \tag{2.21}$$

$$b^{(1)}(\infty) = \frac{w_1 w_3 b_{55}(0) + w_3 w_5 b_{11}(0) + w_5 w_1 b_{33}(0)}{w_1 w_3 + w_3 w_5 + w_5 w_1}$$
$$b^{(2)}(\infty) = \frac{w_2 w_4 b_{66}(0) + w_4 w_6 b_{22}(0) + w_6 w_2 b_{44}(0)}{w_2 w_4 + w_4 w_6 + w_6 w_2} \tag{2.22$'$}$$

if (2.17) is violated, i.e. the $\nu_1, \ldots, \nu_6$ do not obey any of the sets of equations

$$\nu_1 = \nu_3 = \nu_5 \tag{2.23a}$$

or

$$\nu_2 = \nu_4 = \nu_6 \tag{2.23b}$$

or

$$\nu_1 = \nu_2 \qquad \nu_3 = \nu_4 \qquad \nu_5 = \nu_6 \tag{2.23c}$$

or

$$\nu_1 = \nu_4 \qquad \nu_3 = \nu_6 \qquad \nu_5 = \nu_2 \tag{2.23d}$$

or

$$\nu_1 = \nu_6 \qquad \nu_3 = \nu_2 \qquad \nu_5 = \nu_4. \tag{2.23e}$$

Let us now suppose that (2.23c) holds; equations (2.11) then yield the relevant matrix elements (all others vanish at infinity)

$$\dot{b}_{12} + w_1(b_{12} - b_{34}) = 0$$
$$\dot{b}_{34} + w_3(b_{34} - b_{56}) = 0$$
$$\dot{b}_{56} + w_5(b_{56} - b_{12}) = 0.$$

Therefore,

$$b_3(\infty) := b_{12}(\infty) = b_{34}(\infty) = b_{56}(\infty) \qquad w_1 w_3 \dot{b}_{56} + w_3 w_5 \dot{b}_{12} + w_5 w_1 \dot{b}_{34} = 0$$

and

$$b_3(\infty) = \frac{w_1 w_3 b_{56}(0) + w_3 w_5 b_{12}(0) + w_5 w_1 b_{34}(0)}{w_1 w_3 + w_3 w_5 + w_5 w_1}.$$

If we finally assume (2.23d) to hold, we arrive at the case where all ν_i are equal,

$$\nu_i = \nu \qquad i = 1, \ldots, 6.$$

The statistical weights in the averages become all equal, seven independent averages arise and the asymptotic configuration can be written as

$$B|_{t=\infty} \begin{pmatrix} b_1 & b_3 & b_4 & b_5 & b_4^* & b_6 \\ & b_2 & b_6^* & b_7 & b_5^* & b_7^* \\ & & b_1 & b_3 & b_4 & b_5 \\ & & & b_2 & b_6^* & b_7 \\ & \text{h.c.} & & & b_1 & b_3 \\ & & & & & b_2 \end{pmatrix}$$

where

$$b_1 = \tfrac{1}{3}(b_{11}(0) + b_{33}(0) + b_{55}(0))$$
$$b_2 = \tfrac{1}{3}(b_{22}(0) + b_{44}(0) + b_{66}(0))$$
$$b_3 = \tfrac{1}{3}(b_{12}(0) + b_{34}(0) + b_{56}(0))$$
$$b_4 = \tfrac{1}{3}(b_{13}(0) + b_{35}(0) + b_{51}(0))$$

and so on.

2.4. Solutions in general reference frames

In the following, we shall give a preliminary discussion of this problem and study asymptotic solutions in the vicinity of the cyclically symmetric solutions, explored by unitary transformations

Obviously, only the $L = 0$ case survives unitary transformation in an exact sense: we had in the case $L = 0$, not all w_i equal,

$$B(t) \xrightarrow[t\to\infty]{} b^{(0)}_{(\infty)}\mathbb{I}. \tag{2.24}$$

As is equally clear, the symmetries characterized by (L, K) and manifesting themselves as decouplings in the equations of motion do not persist if unitary transformations $U \neq \mathbb{I}$ are performed. Nonetheless, we should expect (and indeed find) traces of these symmetries for 'small rotations'

$$U = \mathrm{e}^{\mathrm{i}\epsilon W} \qquad W^{+} = W \qquad \epsilon\|W\| \ll 1. \tag{2.25}$$

It is hard to make any general statements. If we expand in terms of ϵ we must keep in mind that the expansion limits the long time integration, in particular the limit $t \to \infty$ becomes meaningless. Notwithstanding this difficulty, if we take $\|V\|$ sufficiently large to obtain a rapid approach to this limit we find an intermediate region where the asymptotic B is already sufficiently stable so that an estimate is meaningful: the shift $\Delta B_{jk}|_{t_{\mathrm{large}}}$ of matrix elements due to small unitary rotation can be estimated as

$$\Delta B_{jk}|_{t_{\mathrm{large}}} = \epsilon^2 \Delta B \tag{2.26}$$

where ΔB is a level splitting due to an $L \geqslant 1$ symmetry at $\epsilon = 0$ (for example (2.22′) gives $\Delta B = b^{(1)} - b^{(2)}$).

To get some more detailed insight we had to take recourse to numerical experiments. We need two inputs:

(a) A Hermitian matrix $B|_{t=0}$ whose entries are complex numbers with random real and imaginary parts:

$$-1 \leqslant \mathrm{Re}\, B^{(0)}_{ik} \leqslant 1 \qquad \text{and} \qquad -1 \leqslant \mathrm{Im}\, B_{ik}(0) \leqslant 1 \qquad i, k = 1, \ldots, N. \tag{2.27}$$

(b) A complex matrix V as constructed in (2.8) and (2.7); as input we take random numbers

$$-1 \leqslant \mathrm{Re}\, \nu_i \leqslant 1 \quad \text{and} \quad -1 \leqslant \mathrm{Im}\, \nu_i \leqslant 1. \tag{2.28′}$$

To construct unitary matrices U close to $\mathbb{I}$, controlled by a parameter ϵ we proceed as follows: we take column vectors

$$\tilde{u}_i = e_i + \epsilon h_i \qquad i = 1, \ldots, N$$

where e_i are the standard unit vectors ($e_1 = (1, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots)$ etc) of length N and

$$h_i = (h_{mi}) \qquad m, i = 1, \ldots, N$$

with random entries $-1 \leqslant \mathrm{Re}\, h_{mi} \leqslant 1, -1 \leqslant \mathrm{Im}\, h_{mi} \leqslant 1$. A Gram–Schmidt procedure yields an orthonormal set of linear independent vectors h_i, $i = 1, \ldots, N$

$$\tilde{u}_i \xrightarrow[\text{Gram–Schmidt}]{} u_i$$

for all ϵ, and hence a unitary matrix U.

We then perform a sample of computations varying ε,

$$\varepsilon = 10^{-3}, 10^{-2}, 10^{-1}$$

and compare with the results of a calculation at $\varepsilon = 0$ where the symmetries discussed hold exactly. For each member of the sample the random matrices $B|_{t=0}$ and V are fixed; we furthermore kept the random vectors h_i fixed and varied only ε

Results for $N = 4$, $L = 1$:

$\varepsilon = 0$:	the asymptotic configuration is diagonal and twofold degenerate as predicted by the $L = 1$ symmetry.
$\varepsilon = 10^{-2}, \varepsilon = 10^{-3}$:	the symmetry pattern obtained at $\varepsilon = 0$ remains, a dependence on the parametrization of U is not detected (closeness to the symmetry pattern is essentially controlled by ε).
$\varepsilon = 10^{-1}$:	non-diagonal asymptotic configurations result. The twofold degeneracy of the diagonal elements however persists.

We have also calculated the cases $N = 6$, $L = 1, 2$ and $N = 12$, $L = 1, 2, 3$. The results are qualitatively the same as in the $L = 1$ case.

Nonetheless, we should keep in mind that these 'symmetry regions' are rather small: if we throw a dice to get a complex Lindblad matrix V the probability of obtaining the asymptotic configuration (2.20) and (2.21) (for $N = 6$) or (2.15) for $L = 0$ and all N, is very close to 1. Configurations with non-trivial invariant subspaces corresponding to non-trivial R-cycles will be improbable since the volume (i.e the probability measure for the result of dice throwing) of the region in parameter space around these subspaces determined by ϵ compared to the region (2.28) is very small and decreases with the dimension of our Hilbert space.

3. An algebraic model

As discussed above, the problem we are confronted with is in a way the problem of constructing 'square roots' of positive diagonal matrices V^+V. This reminds us of Dirac's solution of, in essence, a similar problem: the construction of 'square roots' of the d'Alembert operator (which should be taken in a Euclidean metric because of positivity to emphasize the analogy). Dirac's brilliant solution of the problem is intimately connected to the construction of Clifford algebras which can also play a role in our much more restricted context.

The constituent anticommutation relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}\mathbb{I} \qquad i, j = 1, \ldots, r$$

have irreducible representations constructed as $r \times r$ matrices with $r = 2^m$, $m = 2, 4, \ldots$ even. We shall consider the lowest dimensional case $r = 4$. Explicitly, we take the (Euclidean) Dirac representation

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad i = 1, \ldots, 3 \qquad \gamma_4 = \begin{pmatrix} 0 & \mathrm{i}\mathbb{I} \\ -\mathrm{i}\mathbb{I} & 0 \end{pmatrix} \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \qquad \gamma_6 = \mathbb{I}.$$

$\gamma_7 \cdots \gamma_{12}$ are determined as

$$\frac{\mathrm{i}}{2}(\gamma_i \gamma_j - \gamma_j \gamma_i)$$

and $\gamma_{13} \cdots \gamma_{16}$, finally, as

$$\frac{\mathrm{i}}{6}\varepsilon_{ijk\ell}\gamma_j\gamma_k\gamma_\ell$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices, $\mathbb{I}$ denotes 2×2 or 4×4 unit matrices, the $\gamma_1 \cdots \gamma_{16}$ are[2] linear independent and span the 16-dimensional Clifford algebra, as a vector space over $\mathbb{R}$ they span the space of 4×4 Hermitian matrices, as a vector space over $\mathbb{C}$ the space of 4×4 complex matrices is obtained.

We write the Lindblad matrix V as

$$V = \sum v_i \gamma_i \qquad v_i \in \mathbb{C} \tag{3.1}$$

and the observables B as

$$B = \sum f_i(t) \gamma_i \qquad f_i \in \mathbb{R}. \tag{3.2}$$

If we take

$$V = \sum_{i=1}^{16} v_i \gamma_i$$

i.e. if we take V as the most general complex 4×4 matrix, we find from

$$\begin{aligned} \dot{B} &= V^+ B V - \tfrac{1}{2}[V^+ V, B]_+ \\ B &= H, A, \ldots \end{aligned} \tag{2.1$'$}$$

that only the trivial component $f_6(t)$ ($\gamma_6 = \mathbb{I}$) decouples,

$$\dot{f}_6(t) = 0.$$

As a close inspection of the multiplication rules of our representation shows, after all the representation we use is irreducible: We reproduce our statement on multiples of $\mathbb{I}$ being the only non-trivial asymptotic configuration for general V. We have

$$B|_{t=\infty} = f_6(0)\gamma_6$$

(the averaging (2.15) and (2.21) is cut short here by the decomposition into the basis $\{\gamma_i\}$).

In the attempt to alleviate this strict constraint on asymptotic configurations, we look for subalgebras. The simplest one is spanned by the four diagonal matrices $\gamma_5, \gamma_6, \gamma_7, \gamma_{12}$:

$$\Gamma_{\mathrm{diag}} = \left\{ M_{\mathrm{diag}} | M_{\mathrm{diag}} = \sum_{i=1}^{4} c_i \gamma_{K_i}\, c_k \in \mathbb{C} \text{ or } \mathbb{R} \right\} \qquad K = [5, 6, 7, 12].$$

Furthermore, there are three less trivial eight-dimensional subalgebras:

$$\Gamma^{(i)}_{\mathbb{C},\mathbb{R}} = \left\{ M^{(i)} | M^{(i)} = \sum_{k=1}^{8} c_k \gamma_{K^{(i)}_k} c_k \in \text{ or } \mathbb{R} \right\} \qquad K^{(1)} = [3, 4, 15, 16, 5, 6, 7, 12]$$

$$K^{(2)} = [1, 2, 13, 14, 5, 6, 7, 12] \quad K^{(3)} = [8, 9, 10, 11, 5, 6, 7, 12]. \tag{3.3}$$

[2] Explicitly, we use the following enumeration:

$$\gamma_5 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \gamma_6 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \gamma_7 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \gamma_8 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \gamma_9 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$\gamma_7 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\gamma_{10} = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \gamma_{11} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \gamma_{12} = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \gamma_{13} = \begin{pmatrix} 0 & -\mathrm{i}\sigma_1 \\ \mathrm{i}\sigma_1 & 0 \end{pmatrix}$$

$$\gamma_{14} = \begin{pmatrix} 0 & \mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0 \end{pmatrix} \gamma_{15} = \begin{pmatrix} 0 & -\mathrm{i}\sigma_3 \\ \mathrm{i}\sigma_3 & 0 \end{pmatrix} \gamma_{16} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}.$$

It can now be inferred from the multiplication rules of the algebra that the map

$$B \longmapsto V^+BV - \tfrac{1}{2}[V^+V, B]_+$$

decouples

$$\begin{array}{lll} \text{for} \quad V \in \Gamma_{\mathbb{C}}^{(1)} & & \text{the components } f_6 \text{ and } f_7 \\ \text{for} \quad V \in \Gamma_{\mathbb{C}}^{(2)} & & \text{the components } f_6 \text{ and } f_{12} \\ \text{for} \quad V \in \Lambda_{\mathbb{C}}^{(3)} & & \text{the components } f_5 \text{ and } f_6 \end{array}$$

i.e.

$$B|_{t=\infty} = \begin{pmatrix} f_6(0) & + & f_7(0) \\ f_6(0) & - & f_7(0) \\ f_6(0) & - & f_7(0) \\ f_6(0) & + & f_7(0) \end{pmatrix}$$

$$B|_{t=\infty} = \begin{pmatrix} f_6(0) & - & f_{12}(0) \\ f_6(0) & + & f_{12}(0) \\ f_6(0) & + & f_{12}(0) \\ f_6(0) & - & f_{12}(0) \end{pmatrix}$$

$$B|_{t=\infty} = \begin{pmatrix} f_6(0) & + & f_5(0) \\ f_6(0) & + & f_5(0) \\ f_6(0) & - & f_5(0) \\ f_6(0) & - & f_5(0) \end{pmatrix}.$$

We see that the total degeneracy is reduced to a pair degeneracy (by the way this pair degeneracy appears in all possible groupings).

The mechanism for this reduction of degeneracy should be expected to be similar to the case $L = 1$ discussed above: closer inspection of the representation matrices used here shows that a simple transformation of the basis $\{\gamma_i, \mathrm{i}\varepsilon K^{(j)}\}$, $j = 1, 2, 3$ leads to a decoupling of the equation of motion into two decoupled sets with one zero mode each.

The important difference from the cases discussed in the previous section however is that the parameter space used for the construction of the $L = 1$ Lindblad matrix V can now be exactly given, $\mathbb{C}^8$. The parameter space corresponding to the construction described in section 2 was $\mathbb{C}^4$.

The choice of one of the $\Gamma_{\mathbb{C}}^{(i)}$ as a set of Lindblad matrices is the best one can do to obtain reduced degeneracy: if we adjoin any of the remaining basis vectors to get a larger reference space for V we immediately are back to the case of total degeneracy—only f_6 decouples. On the other hand, the case of no degeneracy occurs only if we choose Γ_{diag}—a trivial case.

It should be noted that only diagonal asymptotic (non-trivial) configurations occur in this model.

4. A classification of Lindblad matrices V and the influence of Hamiltonian motion

Suppose we are given a reference frame in an N-dimensional Hilbert space—a basis determined by the physical setting of the system we are to describe—and a Lindblad matrix V. In the first step, we transform to a frame in which V^+V is diagonal with positive diagonal elements $w_i > 0$ (rank $(V) = N$); V is then built up by orthonormal column vectors $\vec{v}_i$,

$$V = (\vec{v}_1, \ldots, \vec{v}_N). \tag{4.1}$$

Now let $\vec{e}_1, \ldots, \vec{e}_N$ denote the usual unit vectors $\vec{e}_1 = (1, 0, 0, \ldots)$, $\vec{e}_2 = (0, 1, 0, 0 \ldots)$ etc, and

$$f : [1, \ldots] \longmapsto [i_1, \ldots, i_N] \tag{4.2}$$

a permutation. It is ancient lore that f can be uniquely decomposed into a set of cycles

$$f = (i_{\alpha 1}, \ldots, i_{\alpha k})(i_{\beta 1}, \ldots, i_{\beta l}) \ldots . \tag{4.3}$$

We now define

$$V_f = (\vec{e}_{i\alpha 1}, \ldots, \vec{e}_{i\beta 1}, \ldots, \vec{e}_{i\alpha k}, \ldots, \vec{e}_{i\beta l}, \ldots) \cdot \Gamma_{\text{diag}} \tag{4.4}$$

where Γ_{diag} is an $N \times N$ diagonal matrix with complex entries (this definition is just another way of writing (2.8): the $\vec{e}_{i\alpha k}$ figures as the αkth column). As we discussed above, to each cycle corresponds an invariant subspace—the corresponding diagonal elements of the observable B moving according to the Lindblad equation decouple $-$, B restricted to this subspace approaches the identity of this subspace (see (2.15) and (2.22) ff). Hence the asymptotic configuration $B|_{t=\infty}$ is invariant under the product of unitary transformations acting in these subspaces,

$$U_{\text{inv}} = U_{c\alpha} \oplus U_{c\beta} \oplus \cdots . \tag{4.5}$$

Thus iff V coincides with an element of the set

$$\mathfrak{V}_{\text{inv}} = \left\{ V | V = U^+_{\text{inv}} V_f U_{\text{inv}},\ \text{all}\, U_{c\alpha}, U_{c\beta}, \Gamma_{\text{diag}} \right\} \tag{4.6}$$

the asymptotic configuration reads

$$B|_{t=\infty} = b^{(\alpha)}(\infty)\mathbb{I}_\alpha \oplus b^{(\beta)}(\infty)\mathbb{I}_\beta \oplus \cdots . \tag{4.7}$$

Two limiting cases are of interest. Consider the case where f is a cyclic permutation (the $L = 0$ case of section 2) and (4.3) consists of one factor and U_{inv} is a unitary transformation in $\mathfrak{H}$. If on the other hand f is the identity we see that V_f is diagonal and U_{inv} is the direct product of $U(1)$ transformation; it is diagonal with phase factors as entries. To decide which of the described categories a given V eventually belongs to we have to probe the elements of $\mathfrak{V}_{\text{inv}}$ starting with this last case and ending with the first—a very 'theoretical' prospect indeed. In by far the most cases, this fictional check would end with the identity in $\mathfrak{H}$ as the corresponding asymptotic configuration as should be clear from the relative 'volumes' of the elements of $\mathfrak{V}_{\text{inv}}$. Let us now include Hamiltonian motion. The observable B obeys the equations of motion,

$$\dot{\tilde{B}} = \tilde{V}^+ \tilde{B} \tilde{V} - \tfrac{1}{2}[\tilde{V}^+ \tilde{V}, \tilde{B}]_+ \tag{4.8}$$

where

$$\tilde{B} = \mathrm{e}^{-\mathrm{i}Ht} B\, \mathrm{e}^{\mathrm{i}Ht}$$

and

$$\tilde{V} = \mathrm{e}^{-\mathrm{i}Ht} V\, \mathrm{e}^{\mathrm{i}Ht}.$$

The case where

$$[H, V^+ V] = 0 \tag{4.9}$$

is particularly simple: we work in the energy basis in $\mathfrak{H}$ in which both H and V^+V are diagonal. Equations (2.12) are seen to remain unchanged so that the asymptotic factors are not influenced by the Hamiltonian. If on the other hand V^+V is not diagonal in the energy basis the asymptotic configurations retain the structure established above. The integrals of motion ((2.14) ff) however no longer hold and the determination of the asymptotic constants $b^{(\alpha)}(\infty)$ is more complicated.

5. States

Up to now we have been working exclusively in the Heisenberg picture. Since the motion of states is of interest in its own right we should add a few remarks on the Schrödinger picture for Lindblad motion and write down the results for the maps

$$\hat{\tau}(V) : \varrho|_{t=0} \mapsto \varrho|_{t=\infty} \tag{5.1}$$

mapping an initial state, the density matrix at $t = 0$, on the final state at large times. In discussing $\hat{\tau}(V)$ we strictly follow the methods described above.

Expectation values in the Heisenberg picture derive from a physically defined state ϱ_0 which we take as the density matrix at $t = 0$, identified with the Schrödinger state at this time. We fix the latter by prescribing a probability distribution.

The expectation value for the observable B in the state ϱ_0 is

$$\begin{aligned} \langle B\rangle_t &= \mathrm{Tr}(B(t)\varrho_0) = \mathrm{Tr}\big(\mathrm{e}^{\vec{L}_B t} B(0)\varrho_0\big) \\ &= \mathrm{Tr}\big(B(0)\varrho_0\,\mathrm{e}^{\overleftarrow{L}_B t}\big) \\ &=: \mathrm{Tr}\big(B(0)\,\mathrm{e}^{\vec{L}_\varrho t}\varrho_0\big) \end{aligned} \tag{5.2}$$

where L_B is read from (1.6)

$$L_B(\cdot) = V^+ \cdot V - \tfrac{1}{2}[V^+V, \cdot]_+ \tag{5.3}$$

the arrow indicates the direction of action; hence

$$L_\varrho(\cdot) = V \cdot V^+ - \tfrac{1}{2}[V^+V, \cdot]_+ \tag{5.4}$$

and the Lindblad equation for states

$$\dot{\varrho} = -\mathrm{i}[H, \varrho] + V\varrho V^+ - \tfrac{1}{2}[V^+V, \varrho] \tag{5.5}$$

reads, for Lindblad matrices derived from (2.8),

$$\dot{\varrho}_{li} = v_{f_L^{-1}(l)} v^*_{f_L^{-1}(i)} b_{f_L^{-1}(l) f_L^{-1}(i)} - \tfrac{1}{2}(|v_i|^2 + |v_l|^2) b_{li}. \tag{5.6}$$

Asymptotic solutions are derived along the lines described in section 2.

In the notation leading to (2.15) we obtain

$$\varrho|_{t=\infty} = (\tilde{\varrho}_{ik}) \quad \varrho|_{t=0} = (\varrho_{ik}(0)) \qquad \tilde{\varrho}_{ik} = \delta_{ik}\varrho_i(\infty)$$

$$\varrho_{j_{\alpha k}}(\infty) = P^{(\alpha)}_{\text{initial}} \frac{1}{w_{j_{\alpha k}}} \frac{1}{\sum_{l=1}^{K} \frac{1}{w_{j_{\alpha l}}}} \tag{5.7}$$

where

$$W^\alpha = \prod_{k=1}^{K} w_{j_{\alpha k}}$$

and

$$P^{(\alpha)}_{\text{initial}} = \sum_{k=1}^{K} \varrho_{j_{\alpha k} j_{\alpha k}}(0). \tag{5.8}$$

The R-cycles of length $K = \frac{N}{L+1}$ are now denoted as

$$R^{(\alpha)} = \big[j_{\alpha_1} \dots j_{\alpha_K}\big] \qquad \alpha = 1, \dots, L+1.$$

Asymptotic configurations turn out diagonal, for arbitrary initial configurations, except the degeneracies described above.

Lindblad motion carries information on the initial state in form of the partial traces $P^{(\alpha)}_{\text{initial}}$; it is of interest to note that all the information on the initial state drops out in the $L = 0$ case (the generic case defined in the summary), as it does in the Heisenberg picture, so that the asymptotic expectation value is given by (2.15) ($L = 0$, $K = N$) or (2.21) ($N = 6$, $L = 0$). The partial traces have the statistical interpretation of sums of probabilities and give the statistical weight of the invariant subspace determined by $R^{(\alpha)}$.

Solutions in general frames, discussed in section 2.4, seen from the point of view of expectation values, now are interpreted in the following manner.

Let V_L be a Lindblad matrix as given by (2.8) and

$$\mathfrak{V} = \{U V_L U^{-1} | U \in \mathfrak{U}\}$$

the class of unitary equivalents of V_L ($\mathfrak{U}$ is the unitary group in N^2 dimensions). Now, if B_U is a solution of

$$\dot{B}_u = V_U^+ B_U V_U - \tfrac{1}{2}\left[V_U^+ V_U, B_U\right] \tag{5.9}$$

with

$$V_U \in \mathfrak{V}$$

and initial conditions

$$B_U(0) = U B_{\text{in}} U^{-1} \tag{5.10}$$

then

$$B_L(t) = U^{-1} B_U(t) U$$

is a solution of

$$\dot{B}_L = V_L^+ B_L V_L - \tfrac{1}{2}\left[V_L^+ V_L, B_L\right]$$

and

$$\begin{aligned} \langle B_U \rangle_t &= \mathrm{Tr}(B_U(t)\varrho_0) \\ &= \mathrm{Tr}(B_L(t) U^{-1} \varrho_o U) \qquad \text{for all} \quad U \in \mathfrak{U}. \end{aligned} \tag{5.11}$$

Hence, to determine expectation values of observables moving with a Lindblad matrix $V_U \in \mathfrak{V}$ we simply have to handle the much easier problem of solving the Lindblad equation for the generating Lindblad matrix V_L with an initial configuration B_{in} (in fact, this problem is asymptotically solved in (2.15)) and to calculate the expectation value (5.10); ϱ_0 is the Schrödinger state at $t = 0$ corresponding to the physical process described with an observable (5.10). In this way we have solved, asymptotically, the Lindblad equation for a large class of Lindblad matrices. Needless to say, there is an obvious counterpart of this derivation for the Schrödinger picture.

6. Summary

The problem posed in this paper is to discuss the motion of observables or states generated by a Lindblad generator. The latter is 'parametrized' by a Hamiltonian H and Lindblad operators V_j. We restrict our discussions to the case of a finite number of levels and thus to N-dimensional Hilbert spaces and $N \times N$ Hermitian Hamiltonian matrices H and general non-singular $N \times N$ complex matrices $V \in \mathbb{C}^{N^2}$, only one matrix V figures in our ansatz for the Lindblad generator (which in the case of finite dimensions is not an assumption of unnatural purport).

At the centre of our investigation are maps which map an initial configuration of an observable B into the configuration at large times ($t \to \infty$) predicted by the Lindblad equation

$$\tau(V) : B|_{t=0} \xrightarrow[\text{Lindblad}]{} B|_{t=\infty}.$$

Our intention is to characterize the dependence of τ on the Lindblad matrices V and ascertain in particular consequences of symmetries of V for $\tau(V)$. Are there specific structures in asymptotic configurations produced by symmetries of V which may serve as a key for the characterization of $\tau(V)$?

In section 4 we described a classification of Lindblad matrices according to characteristics of asymtoptic configurations. The general feature to note is that observables approach asymptotically the identity in whatever invariant subspace (characterized by cycles in the permutation of levels defining a Lindblad process). Exceptions from this rule are determined by degeneracies in the probability distribution fixing the asymptotic configuration.

Apart from this characterization it is sensible to ask the more qualitative question of how symmetry patterns behave under small symmetry breaking, i.e. under small unitary rotations, unitary transformations close to the identity. To this end, we provide numerical experiments in section 2.2 which support the surmise that symmetry is apparent only in a rather small neighbourhood of unity.

It seems reasonable to assume as a generally applicable idea that this neighbourhood can be significantly extended by endowing the linear space of complex $N \times N$ matrices considered up to now with more structure. In a way, the problem we are confronted with when treating the problem of classifying non-Hermitian Lindblad matrices V is to calculate 'square roots' V of positive, Hermitian matrices V^+V, a problem which is the heart of the definition of a Clifford algebra (well known from the Dirac equation invented to describe relativistic spin $\frac{1}{2}$ particles) spanned by 4×4 complex matrices, the γ-matrices. Two things are now to be discovered:

(i) As expected, Lindblad dynamics in the full 16-dimensional algebra will lead to multiples of unity as asymptotic configurations, in general

(ii) The full algebra contains three subalgebras of eight dimensions and one algebra spanned by diagonal γ, the trivial case. Lindblad dynamics in the eight-dimensional subalgebras leads to asymptotic configurations with two degenerate doublets (the three subalgebras yield exactly the three possibilities of arranging two doublets on four positions). Expressed in the terminology of the general case, this is the $N = 4$, $L = 1$ symmetry which now persists in the whole parameter space $\mathbb{C}^8$. We have thus constructed a model where a non-trivial symmetry is not restricted to small neighbourhoods of $\mathbb{C}^4$ of symmetry configurations (as it was for the situation discussed in section 2.4) but rather holds in all of the parameter space. If we extend the span of any of the three eight-dimensional subalgebras by one more basis vector $\hat{\gamma}$ the asymptotic configuration of the obtained system collapses to a multiple of unity in the new parameter space

For further evaluation of the physical relevance of these results, we should add the following more general possibility of establishing Lindblad dynamical systems. We now consider V as a matrix of matrices and take it as consisting of matrix elements $v_i\mathbb{I}_n$ where $v_i \in \mathbb{I}$ and $\mathbb{I}_n$ is the $n \times n$ unit matrix, N is now an integer multiple of n. The equations of motion (2.11) or (2.12) are now identical in form if the b_{ei} are now considered as $n \times n$ submatrices of the Hamiltonian matrix (non-diagonal in general, of course) or any other observable. The line of arguments leading to the asymptotic configuration is still the same and we derive the following asymptotic forms for, say, the Hamiltonian:

$L=0$: the asymptotic Hamiltonian ($N \times N$ matrix) is the direct sum of N/n identical Hamiltonians ($n \times n$ matrices) obtained from the statistical average (2.15) or, for $N=6$, (2.21);

$L=1$: the asymptotic Hamiltonian is the direct sum of $N/2n$ identical pairs of Hamiltonians ($N/2n$ is assumed integer) obtained from (2.15) as statistical averages of Hamiltonians which are now $n \times n$ submatrices arranged along the diagonal at $t=0$;

and so on.

Physically speaking, for an L such that $N/(L+1)$ is an integer multiple of $n \in \mathbb{N}$, the system defined at $t=0$ disintegrates, for $t \to \infty$, into $(L+1)$ independent, i.e. non-interacting quantum systems: the asymptotic configuration of the Hamiltonian defined at $t=0$ is represented as the direct sum of tne $n \times n$ sub-Hamiltonians obtained from averages (2.15).

An interesting possibility for model building is to include the Lindblad matrix in the set of dynamical variables: taking $\tilde{V}$ as one of the quantities $\tilde{B}$ we find

$$\dot{\tilde{V}} = \tfrac{1}{2}[\tilde{V}^+, \tilde{V}]\tilde{V}$$

a non-linear equation for $\tilde{V}$. Determining the asymptotic solutions of these equations, we find that the asymptotic degeneracy structure observed for B or $\tilde{B}$ is now enforced for V^+V. Degeneracies in V^+V and hence in V then entail the kind of off-diagonal structures defined in equation (2.17 ff) or in the $N=6$ example discussed in section 2.2.

Let us finally turn to the question of what happens if more than one operator V_i is required to parametrize the equation of motion (1.6). The prevalent situation (in the sense of discussion) is that in which the asymptotic configuration is proportional to the unit operator $\mathbb{I}$: the appearance of more than one such operator will simple modify the statistical average (2.15) or (2.21). An interesting more general case conceivably occurs if V_i with nested symmetries can be applied in a specific physical situation. An especially simple case arises if the dynamical variables are embedded in a group manifold or an algebraic structure as in the Clifford algebra example discussed above where the decoupling structure of asymptotic configurations is a matter of symmetry extended over all the available parameter space (cf (3.3) ff). In our example, this means that an arbitrary collection of $V_j \in T^{(i)}_{\mathbb{R},\mathbb{C}}$ yields a direct sum of asymptotic configurations (3.3) ff. This result can be readily extended to higher dimensions $d = 2^{4m}$, $m \in \mathbb{N}$.

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References

[1] Haag R and Kastler D 1964 *J. Math. Phys.* **5** 848
[2] Davies E B 1969 *Commun. Math. Phys.* **11** 277
[3] Hellwig K E and Kraus K 1969 *Commun. Math. Phys.* **11** 214
[4] Hellwig K E and Kraus K 1970 *Commun. Math. Phys.* **16** 142
[5] Kraus K 1971 *Ann. Phys., NY* **64** 311
[6] Stinespring W F 1955 *Proc. Am. Math. Soc.* **6** 211

[7] For a review see Alicki R *Proc. 38th Karpacz International School of Theoretical Physics* at press
[8] Haake F 1973 *Statistical Treatment of Open Systems by Generalised Master Equations* (Berlin: Springer)
[9] Davies E B 1974 *Commun. Math. Phys.* **39** 91
[10] Gorini V and Kossakowski A 1976 *J. Math. Phys.* **17** 1298
[11] Dumcke R 1985 *Commun. Math. Phys.* **97** 331
[12] Strunz W T, Haake F and Braun D Universality of decoherence in the macroworld *Phys. Rev.* at press
[13] Lindblad G 1976 *Commun. Math. Phys.* **48** 119
Davies E B 1974 *Commun. Math. Phys.* **39** 91